APPENDIX TO CHAPTER 8

Cost of production: the isoquant–isocost approach

In this appendix, we develop a model to determine how a profit-maximising firm will combine resources to produce a particular amount of output. The quantity of output that can be produced with a given amount of resources depends on the existing *state of technology*, which is the prevailing knowledge of how resources can be combined. Therefore, we will begin by considering the technological possibilities available to the firm.

The production function and economic efficiency

The ways in which resources can be combined to produce output are summarised by a firm's production function. The *production function* identifies the maximum quantities of a particular good (or service) that can be produced per time period with various combinations of resources and with a given state of technology. The production function can be presented as an equation, a graph or a table.

The production function summarised in Table 8A.1 reflects, for a hypothetical firm, the output resulting from particular combinations of resources. This firm uses only two resources: capital and labour. The amount of capital used is listed in the left-hand column of the table, and the amount of labour employed is listed across the top. For example, if 1 unit of capital is combined with 7 units of labour, the firm can produce 290 units of output per period.

We assume that the firm produces the maximum possible output given the combination of resources employed, and that the same output could not be produced with fewer resources. Since we assume that the production function combines resources efficiently, 290 units is the most that can be produced with 7 units of labour and 1 unit of capital. Thus, we say that the firm's production is technologically (or productively) efficient.

We can examine the effects of adding additional labour to an existing amount of capital by starting with some level of capital and reading across the table. For example, when 1 unit of capital and 1 unit of labour are employed, the firm produces 40 units of output per period. If the amount of labour is increased by 1 unit and the amount of capital employed is held constant, output increases to 90 units, so the marginal product of labour is 50 units. If the amount of labour employed increases from 2 to 3 units, other things constant, output goes to 150 units, yielding a marginal product of 60 units. By reading across the table, you will discover that the marginal product of labour first rises, showing increasing marginal returns from the variable resource (labour), and then declines, showing diminishing marginal returns. Similarly, by holding the amount of labour employed to 1 unit and following down the column, you will find that the marginal product of capital also reflects first increasing marginal returns and then diminishing marginal returns.

TABLE 8A.1 A firm's production function employing units of labour (L) and capital (K)

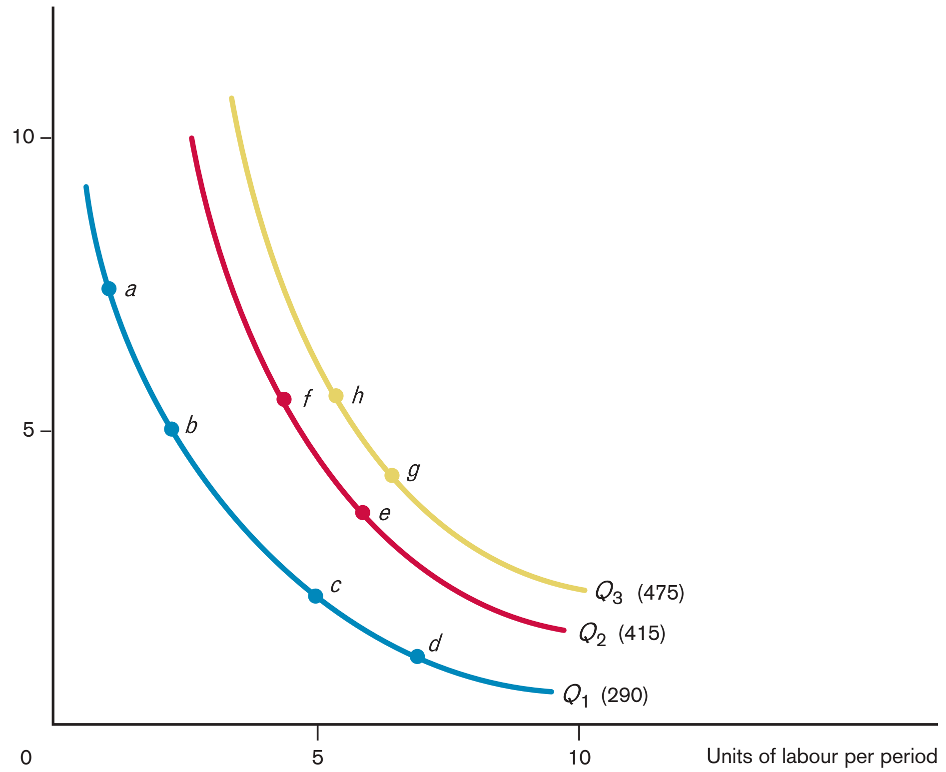
Units of Capital Employed per Period	Units of Labour Employed per Period						
	1	2	3	4	5	6	7
1	40	90	150	200	240	270	290
2	90	140	200	250	290	315	335
3	150	195	260	310	345	370	335
4	200	250	310	350	385	370	390
5	240	290	345	385	420	450	475
6	270	320	375	415	450	475	495
7	290	330	390	435	470	495	510

Isoquants

Notice from the tabular presentation of the production function in Table 8A.1 that different combinations of resources may yield the same level of output. For example, several combinations of labour and capital yield 290 units of output. Some of the information provided in Table 8A.1 can be presented more clearly in graphical form. In Figure 8A.1, the quantity of labour employed is measured along the horizontal axis and the quantity of capital is measured along the vertical axis. The combinations that yield 290 units of output are presented in the figure as points *a*, *b*, *c* and *d*. These points can be connected to form an *isoquant*, Q_1, which shows the possible combinations of the two resources that produce 290 units of output. Likewise, Q_2 shows combinations of inputs that yield 415 units of output, and Q_3 shows combinations that yield 475 units of output. (The colours of the isoquants match those of the corresponding entries in the production function table in Table 8A.1.)

An isoquant, such as Q_1 in Figure 8A.1, is a curve that shows all the technologically efficient combinations of two resources, such as labour and capital, that produce a certain amount of output. *Iso* is from the Greek word meaning 'equal', and *quant* is short for 'quantity'; so *isoquant* means 'equal quantity'. Along a particular isoquant, such as Q_1, the amount of output produced remains constant, in this case 290 units, but the combination of resources varies. To produce a particular level of output, the firm can employ resource combinations ranging from capital-intensive combinations (much capital and little labour) to labour-intensive combinations (much labour and little capital).

For example, a paving contractor can put in a new driveway with ten workers using shovels and hand-rollers; the same job can also be done with only two workers, a road grader and a paving machine. A Saturday-afternoon charity car wash to raise money to send the school band on a Gold Coast holiday at Nara Sea World is labour-intensive, involving perhaps a dozen workers per car. In contrast, a professional car wash is fully automated, requiring only



Isoquant Q_1 shows all technologically efficient combinations of labour and capital that can be used to produce 290 units of output. Isoquant Q_2 is drawn for 415 units, and Q_3 for 475 units. Each isoquant has a negative slope and is convex to the origin.

FIGURE 8A.1 An isoquant map

one worker to turn the machine on and off and collect the money. An isoquant shows such alternative combinations of resources that produce the same level of output. Let us consider some of the main properties of isoquants.

Isoquants further from the origin represent greater output levels

Although we have included only three isoquants in Figure 8A.1, there is a different isoquant for every quantity of output depicted in Table 8A.1. Indeed, there is a different isoquant for every output level the firm could possibly produce, with isoquants further from the origin indicating higher levels of output.

Isoquants slope down to the right

Along a given isoquant, the quantity of labour employed is inversely related to the quantity of capital employed, so isoquants have negative slopes.

Isoquants do not intersect

Since each isoquant refers to a specific level of output, no two isoquants intersect, for such an intersection would indicate that the same combination of resources could, with equal efficiency, produce two different amounts of output.

Isoquants are usually convex to the origin

Finally, isoquants are usually convex to the origin, meaning that the slope of the isoquant gets flatter down along the curve. To understand why, keep in mind that the slope of the isoquant measures the ability of additional units of one resource – in this case, labour (L) – to substitute in production for another – in this case, capital (K). As we said, the isoquant has a negative slope. The slope of the isoquant is the marginal rate of technical substitution (or MRTS), defined between any two resources as:

$$MRTS_{LK} = \frac{-\Delta K}{\Delta L}$$

Here, the $MRTS_{LK}$ indicates the rate at which additional units of labour (ΔL) can be substituted for fewer units of capital ($-\Delta K$) while keeping output (TP) constant. When much capital and little labour are used, the marginal productivity of labour is relatively great and the marginal productivity of capital is relatively small, so one unit of labour will substitute for a relatively large amount of capital.

Consider the case of moving from point a to b along isoquant Q_1 in Figure 8A.1. One unit of labour substitutes for 2 units of capital, so the $MRTS_{LK}$ between points a and b equals 2. But as more units of labour and fewer units of capital are employed, the marginal product of labour declines and the marginal product of capital increases, so it takes more labour to make up for a reduction in capital. For example, in moving from point c to point d in Figure 8A.1, 2 units of labour substitute for 1 unit of capital; hence, the $MRTS_{LK}$ between points c and d equals 1/2.

The extent to which one input substitutes for another, as measured by the marginal rate of technical substitution, is directly linked to the marginal productivity of each input. For example, between points a and b, 1 unit of labour replaces 2 units of capital, yet output remains constant. Thus, labour's marginal product, MP_L – that is, the additional output resulting from an additional unit of labour – must be twice as large as capital's marginal product, MP_K. In fact:

All along the isoquant, the marginal rate of technical substitution of labour for capital equals the marginal product of labour divided by the marginal product of capital, which also equals the absolute value of the slope of the isoquant.

Thus, we can say that:

$$\text{Slope of isoquant} = MRTS_{LK} = \frac{MP_L}{MP_K}$$

where the vertical lines on either side of 'Slope of isoquant' mean the absolute value. For example, between points a and b the slope equals –2, which has an absolute value of 2, which equals the marginal rate of substitution of labour for capital and the ratio of marginal productivities.

If labour and capital were perfect substitutes in production, the rate at which labour substituted for capital would remain fixed along the isoquant, so the isoquant would be a downward-sloping straight line. Since most resources are *not* perfect substitutes, however, the rate at which one substitutes for another changes along an isoquant. As we move down along an isoquant, more labour is required to offset each 1-unit decline in capital, so the slope of the isoquant gets flatter, yielding an isoquant that is convex to the origin.

Let us summarise the main properties of isoquants:

1 Isoquants further from the origin represent greater levels of output.
2 Isoquants slope downward.

3 Isoquants never intersect.
4 Isoquants tend to be convex — that is, bowed towards the origin.

Isocost lines

Isoquants graphically illustrate a firm's production function for all quantities of output the firm could possibly produce. Given these isoquants, how much should the firm produce? More specifically, what is the firm's profit-maximising level of output? The answer depends on the cost of resources and on the amount of money the firm plans to spend. Assume a unit of labour costs the firm $15 000 per year, and the cost for each unit of capital is $25 000 per year. The total cost (*TC*) of production is:

$$TC = (w \times L) + (r \times K)$$
$$= \$15\,000\ L + \$25\,000\ K$$

where w is the annual wage rate, L is the quantity of labour employed, r is the annual cost of capital, and K is the quantity of capital employed. An isocost line identifies all combinations of capital and labour the firm can hire for a given total cost. Again, *iso* is from the Greek word meaning 'equal', so an *isocost* line is a line representing equal total cost to the firm. In Figure 8A.2, for example, the line TC = $150 000 identifies all combinations of labour and capital that cost the firm a total of $150 000. The entire $150 000 could pay for 6 units of capital per year; if the entire budget is spent only on labour, 10 workers per year could be hired; or the firm can employ any combination of resources along the isocost line.

Recall that the slope of any line is the vertical change between two points on the line divided by the corresponding horizontal change (the rise over the run). At the point where the isocost line meets the vertical axis, the quantity of capital that can be purchased equals the total cost divided by the annual cost of capital, or TC/r. At the point where the isocost line meets the horizontal axis, the quantity of labour that can be hired equals the firm's total cost divided by the annual wage, or TC/w. The slope of any isocost line in Figure 8A.2 can be calculated by considering a movement from the vertical intercept to the horizontal intercept. That is, we divide the vertical change ($-TC/r$) by the horizontal change (TC/w), as follows:

$$\text{Slope of isocost line} = -\frac{-TC/r}{TC/w} = -\frac{w}{r}$$

The slope of the isocost line equals minus the price of labour divided by the price of capital, or $-w/r$, which indicates the relative prices of the inputs. In our example, the absolute value of the slope of the isocost line equals w/r, or:

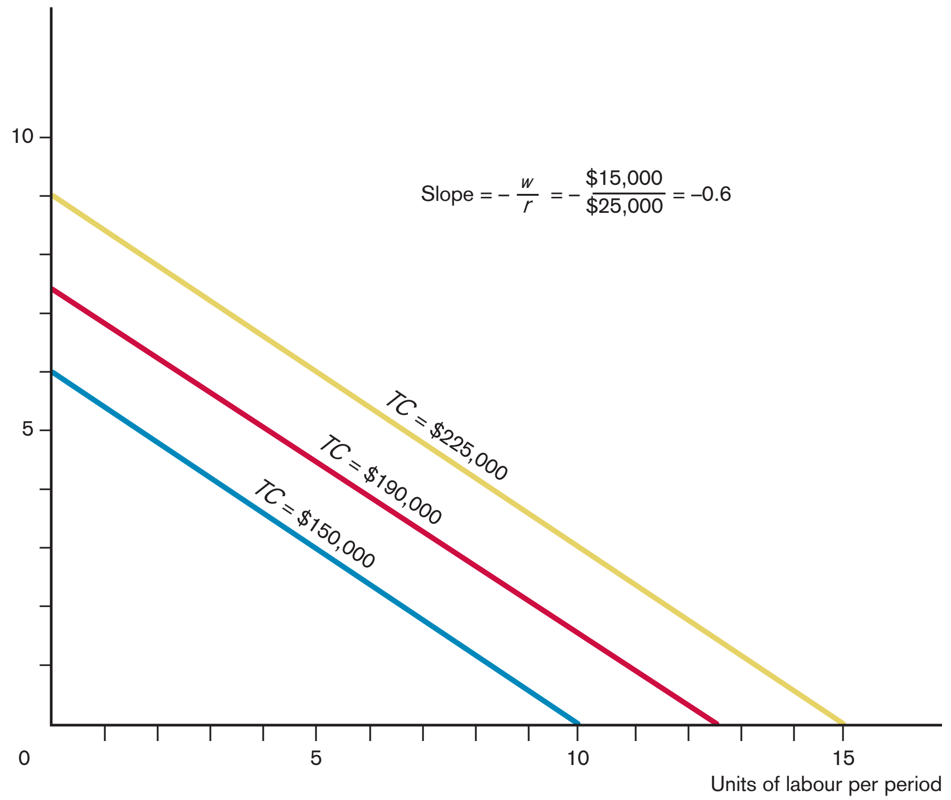
$$\text{Slope of isocost line} = w/r$$
$$= 15\,000/25\,000$$
$$= 0.6$$

The wage rate of labour is 0.6 of the annual cost of capital, so hiring one more unit of labour, without incurring any additional cost, implies that the firm must employ 0.6 units less capital.

A firm is not confined to a particular isocost line. Thus, a firm's total cost depends on how much the firm plans to spend. This is why in Figure 8A.2 we include three isocost lines, not just one, each corresponding to a different total budget. In fact, there is a different isocost line for every possible budget.

These isocost lines are parallel because each reflects the same relative resource price.

Resource prices are assumed to be constant regardless of the amount employed.



Each isocost line shows combinations of labour and capital that can be purchased for a fixed amount of total cost. The slope of each is equal to minus the wage rate divided by the rental rate of capital. Higher levels of cost are represented by isocost lines further from the origin.

FIGURE 8A.2 The firm's isocost lines

The optimal choice of input combinations

We bring the isoquants and the isocost lines together in Figure 8A.3. Suppose the firm has decided to produce 415 units of output and wants to minimise its total cost. The firm could select point *f*, where 6 units of capital are combined with 4 units of labour. This combination, however, would cost $210 000 at prevailing prices. Since the profit-maximising firm wants to produce its chosen output at the minimum cost, it tries to find the isocost line closest to the origin that still touches the isoquant. Only at a point of tangency does a movement in either direction along an isoquant shift the firm to a higher cost level. Hence, it follows that:

The point of tangency between the isocost line and the isoquant shows the minimum cost required to produce a given output.

Consider what is going on at the point of tangency. At point *e* in Figure 8A.3, the isoquant and the isocost line have the same slope. As mentioned already, the absolute value of the slope of an isoquant equals the marginal rate of technical substitution between labour and capital, and the absolute value of the slope of the isocost line equals the ratio of the input prices. So, when a firm produces output in the least costly way, the marginal rate of technical substitution must equal the ratio of the resource prices, or:

$$MRTS_{LK} = MP_L/MP_K = w/r = \$15\,000/\$25\,000 = 0.6$$

This equality shows that the firm adjusts resource use so that the rate at which one input can be substituted for another in production – that is, the marginal rate of technical substitution – equals the rate at which one resource can be traded for another in resource markets, that is the resource price ratio w/r.

If this equality does not hold, it means that the firm could adjust its input mix to produce the same output for a lower cost.

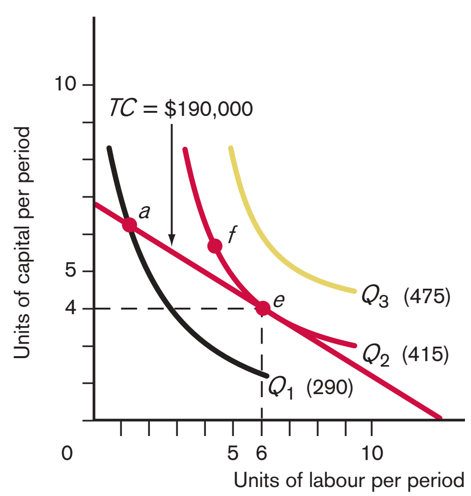
Finally, to demonstrate the consistency between the golden rule for consumer equilibrium (compare Chapter 7) and the producer's equivalent least-cost input combination rule, consider again the firm's input equilibrium condition above:

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{w}{r}$$

Now, simply cross-multiply the wage rate w to the denominator and the MP_K to the numerator of their respective opposite sides to yield:

$$\frac{MP_L}{w} = \frac{MP_K}{r}$$

This is the least-cost input rule for firms operating in competitive resource markets – that is, employ a combination of resources such that *the marginal product per dollar spent is equated across all resources used*. Here only two resources, capital and labour, are employed. If this least-cost input condition is not met, then, assuming the eventual onset of diminishing

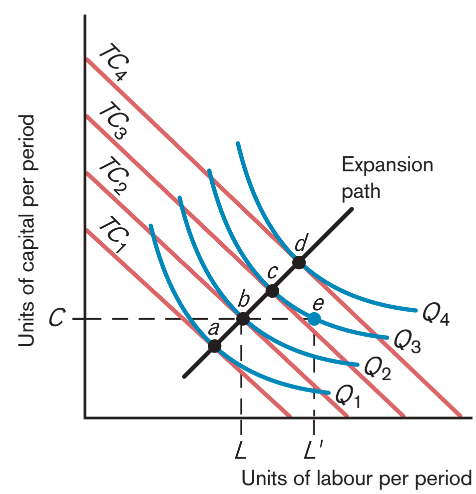


At point *e*, isoquant Q_2 is tangential to the isocost line. The optimal combination of inputs is 6 units of labour and 4 units of capital. The maximum output that can be produced for $190 000 is 415 units. Alternatively, point *e* determines the minimum-cost way of producing 415 units of output.

FIGURE 8A.3 Optimal combinations of inputs returns to variable resources, it is possible to reallocate the amount of resource use between capital and labour until this equilibrium condition does hold.

The expansion path

Imagine a set of isoquants representing each possible level of output. Given the relative cost of resources, we could then draw isocost lines to determine the optimal combination of resources for producing each level of output. The points of tangency in Figure 8A.4 show the least-cost input combinations for producing several output levels.



The points of tangency between isoquants and isocost lines each show the least expensive way of producing a particular level of output. Connecting these tangency points gives the firm's expansion path.

FIGURE 8A.4 The long-run expansion path

For example, the output level Q_2 can be produced at its least-cost combination by employing K units of capital and L units of labour. The line formed by connecting these tangency points is the firm's *expansion path*. If the resources are capital and labour, we often refer to this path as the long-run expansion path. The expansion path need not be a straight line, though it will generally slope upwards, implying that firms will expand the use of both resources in the long run as output increases. Note that we have assumed that the prices of inputs remain constant as the firm varies output along the expansion path, so the isocost lines at the points of tangency are parallel – that is, they have the same slope.

The firm's expansion path indicates the lowest long-run total cost for each level of output. For example, the firm can produce output level Q_2 for TC_2, output level Q_3 for TC_3, and so on. Similarly, the firm's long-run average cost curve conveys, at each level of output, the total cost divided by the level of output. The firm's expansion path and its long-run average cost curve represent alternative ways of portraying costs in the long run, given resource prices and technology.

We can use Figure 8A.4 to distinguish between short-run adjustments in output and long-run adjustments. Let's begin with the firm producing Q_2 at point b, which requires K units of capital and L units of labour. Now, suppose that in the short run, the firm wants to expand output to Q_3.

Since capital is fixed in the short run, the only way to expand output to Q_3 is by expanding the quantity of labour employed to L', which requires moving to point *e* in Figure 8A.4. Point *e* is not the cheapest way to produce Q_3 in the long run, for it is not a tangency point. In the long run, capital usage is variable, and if the firm wishes to produce Q_3, it should adjust capital and shift from point *e* to point *c*, thereby minimising the total cost of producing Q_3.

One final point: if the relative prices of resources change, the least-cost combination of those resources will also change, so the firm's expansion path will change. For example, if the price of labour increases, capital becomes cheaper relative to labour. The efficient production of any given level of output will therefore call for less labour and more capital. With the cost of labour higher, the firm's total cost for each level of output rises. Such a cost increase would also be reflected by an upward shift in the average total cost curve.

SUMMARY

A firm's *production function* specifies the relationship between resource use and output, given prevailing technology. An *isoquant* is a curve that illustrates the possible combinations of resources that will produce a particular level of output. An *isocost* line presents the combinations of resources the firm can employ, given resource prices and the amount of money the firm plans to spend.

For a given level of output – that is, for a given isoquant – the firm minimises its total cost by choosing the lowest isocost line that just touches, or is tangential to, the isoquant. The least-cost combination of resources will depend on the productivity of resources and the relative cost of resources.

REVIEW QUESTIONS

1 (**Properties of Isoquants**) Why are isoquants always negatively sloped in their technically efficient range? Why is it logically impossible for two technically efficient isoquants to intersect?

2 (**Least-cost Rule**) In plain English, explain to your employer what achieving the least-cost input rule means to an expanding firm.

PROBLEM SET

1 (**Input Equilibrium Condition**) Assume that a certain firm is in input equilibrium using given quantities of labour and capital. If the wage rate of labour is w = \$10 per hour and the rental of capital is r = \$25 per hour, and if, in addition, you know that the marginal product of labour MPL = 40 units, what is the value of MPK?

2 (**Optimal Choice of Input Combinations**) Assume that a firm's cost of labour is \$10 per unit and its cost of capital is \$40 per unit.

- **a** Construct an isocost line such that total cost is constant at \$200.
- **b** If this firm is producing efficiently, what is the marginal rate of technical substitution between labour and capital?
- **c** Prove your answer to part (b) using isocost lines and isoquant curves. How are the expansion path and the long-run average cost curve related?